Jacobian varieties and the Torelli theorem

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1 Assumptions and conventions

In this talk, C will be a nonsingular projective curve of genus g over a field k. (Often, we will specialize to the case $k = \mathbb{C}$. Milne states the theorem for k perfect, but I'm told that it holds over all k.) We fix a base point $p \in C(k)$.

2 The Jacobian variety

Idea: the Jacobian variety Jac of C parametrizes degree-0 line bundles on C.

There is a natural way to give Jac the structure of a scheme, and in fact it is an abelian variety of dimension g. (Historically, this was essentially the original definition of g!) In the case $k = \mathbb{C}$, we can view Jac as a complex torus: $\mathrm{Jac} = \mathbb{C}^n / \Lambda$ where $\Lambda \cong \mathbb{Z}^{2g}$ is a lattice.

Aside: moduli description of Jac. We want $\mathrm{Jac}(k) = \mathrm{Pic}^0(C)$. We will turn this into a functor that we hope will be represented by Jac. The naive way to define our functor would be to demand that the functor of "T-points" of Jac (i.e. morphisms $T \to \mathrm{Jac}$) is given by $\mathrm{Pic}^0(C \times_k T)$. But this is bad: even on a curve like $\mathbb{P}^1$ with $\mathrm{Pic}^0(\mathbb{P}^1) = 0$ (so we want $\mathrm{Jac} = \mathrm{Spec}\, k$), it can easily happen that $\mathrm{Pic}^0(C \times_k T)$ is nontrivial. To fix this, declare that the T-points correspond naturally to $\mathrm{Pic}^0(C \times_k T)/q^* \mathrm{Pic}^0(T)$, where q is the natural map $C \times_k T \to T$. It can be shown that this functor is representable by an abelian variety, provided that C has a k-point.

Given the base point p, we have a natural map $\phi : C \to \mathrm{Jac}$ defined by $\phi(q) = \mathcal{O}(q - p)$. This is called the Abel-Jacobi map; it is a closed embedding for $g \geq 1$ and an isomorphism for $g = 1$.

We can soup the Abel-Jacobi map up to give us a map $C^r \to \mathrm{Jac}$, for any r, which is defined by $(x_1, \ldots, x_r) \mapsto \phi(x_1) + \cdots + \phi(x_r)$, using the group law of Jac. In fact, this is invariant under permutations of the coordinates, so it factors through the rth symmetric power $S^r C$ of C. We call this $\phi^{(r)} : S^r C \to \mathrm{Jac}$, and call its image W^r.

Abel's theorem: for points $x_1, \ldots, x_r, y_1, \ldots, y_r$, the divisors $\sum x_i$ and $\sum y_i$ are linearly equivalent if and only if $(x_1, \ldots, x_r)$ and $(y_1, \ldots, y_r)$ map to the same point in Jac. Equivalently, the fibers of $\phi^{(r)}$ correspond exactly to the linear equivalence classes of (effective) degree-r divisors.

Let's look at some examples. The simplest example is $\mathbb{P}^1$. This, of course, doesn't have any nontrivial degree-0 line bundles, so $\mathrm{Jac} = \operatorname{Spec} k$. The simplest nontrivial example is where C is an elliptic curve, in which case $\phi : C \to \mathrm{Jac}$ is an isomorphism. Next, consider the case $g = 2$. Here, C is a double cover of $\mathbb{P}^1$ branched at 6 points. Something interesting happens with $\phi^{(2)}$ here. From the double cover $C \to \mathbb{P}^1$, we get a family of degree-2 divisors, parametrized by $\mathbb{P}^1$, which are all linearly equivalent to each other. By Abel's theorem, these all map to the same point in Jac. In fact, $\phi^{(2)} : S^2C \to \mathrm{Jac}$ is exactly the blow-down along this divisor. This generalizes: for any g, the map $\phi^{(g)} : S^gC \to \mathrm{Jac}$ is birational.

3 The Torelli theorem

Theorem (Torelli, 1914-15): the Jacobian, together with the data of a principal polarization, determines C up to isomorphism.

The definition of a polarization depends somewhat on what sect of algebraic geometry you belong to. To most modern people, a polarization is a choice of isogeny $\lambda : \mathrm{Jac} \to \mathrm{Jac}^\vee$ that is compatible with the dual map $\mathrm{Jac} = \mathrm{Jac}^{\vee\vee} \to \mathrm{Jac}^\vee$; a polarization is principal if it is an isomorphism. Given our Abel-Jacobi map $\phi : C \to \mathrm{Jac}$, it turns out that there is a canonical choice of principal polarization $\mathrm{Jac} \to \mathrm{Jac}^\vee = \mathrm{Pic}^0(\mathrm{Jac})$, defined on points by:

$$a \mapsto t_a^*\mathcal{O}(W^{g-1}) \otimes \mathcal{O}(-W^{g-1}). \tag{1}$$

Here, $t_a : \mathrm{Jac} \to \mathrm{Jac}$ is translation by a.

On the other hand, more classical people (working over $\mathbb{C}$) often define a polarization of the abelian variety $X = \mathbb{C}^g/\Lambda$ to be a positive definite Hermitian form H on $\mathbb{C}^g$ whose imaginary part takes integer values on $\Lambda \times \Lambda$. Griffiths and Harris ([1]) defines a polarization to be the first Chern class of a Hodge form ω, $[\omega] \in H^2_{\mathrm{sing}}(C, \mathbb{Z})$.

Milne ([2], 122-125) gives an elementary proof involving the combinatorics of the various subvarieties $W^r \subset \mathrm{Jac}$. Mumford ([3]) also gives sketches of four different ways to approach the theorem.

4 Applications

Here's a modern application of Torelli's theorem, courtesy of Alex Youcis. Let C be a curve over a number field K. We claim that only finitely many non-isomorphic curves C can have first ℓ-adic étale cohomology groups $H^1_{\text{ét}}(C, \mathbb{Q}_\ell)$ isomorphic to each other as G_K-modules. The proof works in three steps. First, it can be shown that $H^1(C, \mathbb{Q}_\ell)$ is naturally isomorphic to the dual of the Tate module of the Jacobian, $(V_\ell \mathrm{Jac})^\vee$. Next, a theorem of Faltings (and Tate?) states that over a number field, abelian varieties are determined up to isogeny by their Tate modules, and moreover that there are only finitely many abelian varieties in any given isogeny class. Finally, there are only finitely many possible Jacobians, so by Torelli there are only finitely many possible curves!

5 Further results and problems

In a related direction, the Schottky problem asks which abelian varieties arise as Jacobians of curves. In moduli terms: we have a map from the moduli space of curves to the moduli space of abelian varieties equipped with principal polarizations. What is its image? This is still not resolved, although there are partial results.

References

[1] Phillip Griffiths and Joseph Harris, *Principles of Algebraic Geometry*, 1978.

[2] J.S. Milne, *Abelian Varieties* (online notes), 2008.

[3] David Mumford, "Curves and their Jacobians" (appendix to *The Red Book of Varieties and Schemes*), 1975, reprinted 1999.

[4] Alex Youcis, "Some examples of geometric Galois representations", 2015. https://ayoucis.wordpress.com/2015/01/26/some-examples-of-geometric-galois-representations/.